Improvements to a Branch-Cut-and-Price Algorithm for the Exact Solution of Parallel Machines Scheduling Problems

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Outline

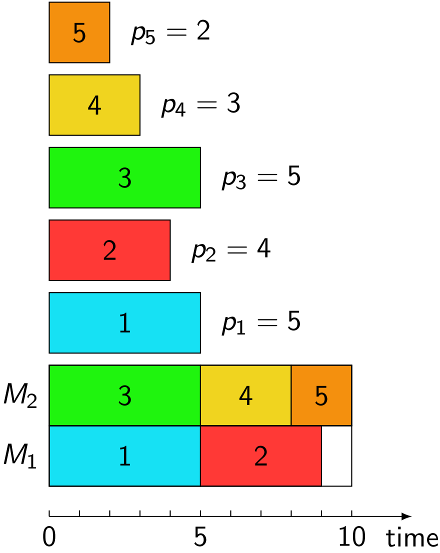
1. The Parallel Machines Scheduling Problem
2. The BCP of Pessoa, Uchoa, Poggi, Rodrigues (2010)
3. The Improved Algorithm
   - Newly Proposed Cuts over Extended Variables
   - Additional Known Cuts
   - Alternative Time-Indexed Formulations
   - New Cuts over TIF
4. Experiments
5. Conclusions

The Parallel Machines Scheduling Problem

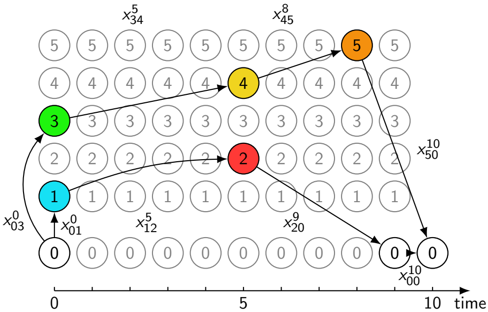
- $J = \{1, \ldots, n\}$
- $M = \{1, \ldots, m\}$
- Processing times p_j
- Cost $f_j(C_t)$

Weighted Tardiness:

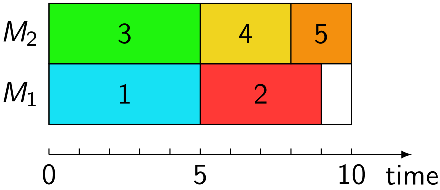
- Due dates d_j
- Weights w_j
- Minimize $\sum w_j T_j$



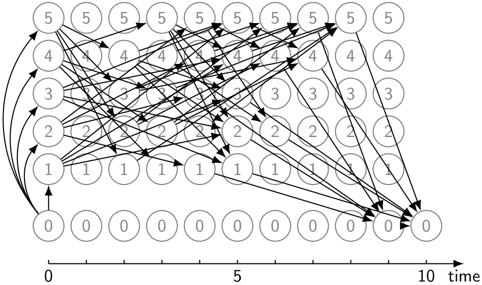
The Arc-Time-indexed Formulation



- Variables x_{ij}^{t}: job j succeeds job i at time t
- Schedules are paths in $G = (V, A)$
  - $V = \{(i, t)\}$
  - $A = \{((i, t - p_i), (j, t))\}$
  - $i, j \in J_0 = J \cup \{0\}$
  - $t \in \{0, \dots, T\}$

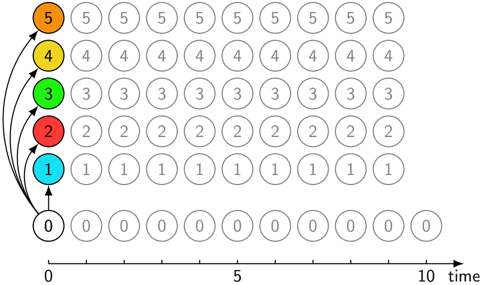


The Arc-Time-indexed Formulation



$$\min \sum_{i \in J_0} \sum_{j \in J \setminus \{i\}} \sum_{t=p_i}^{T-p_j} f_j\,(t+p_j)\,x_{ij}^t$$

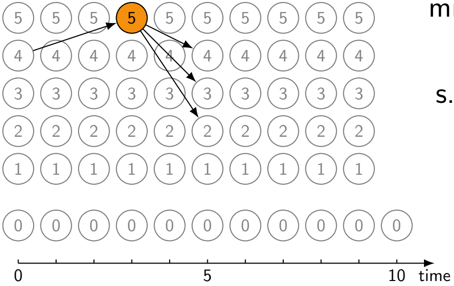
The Arc-Time-indexed Formulation



$$\min \sum_{i \in J_0} \sum_{j \in J \setminus \{i\}} \sum_{t=p_i}^{T-p_j} f_j\left(t+p_j\right) x_{ij}^{t}$$

$$\text{s.t.} \quad \sum_{j \in J_0} x_{0j}^{0} = m$$

The Arc-Time-indexed Formulation

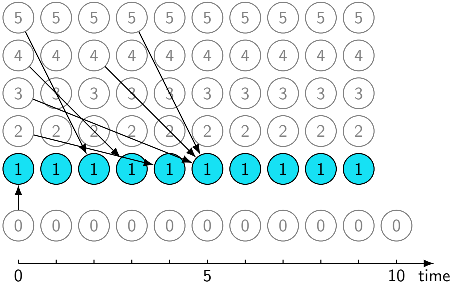


$$\min \sum_{i \in J_0} \sum_{j \in J \setminus \{i\}} \sum_{t=p_i}^{T-p_j} f_j\left(t+p_j\right) x_{ij}^t$$

$$\text{s.t.} \sum_{\substack{j \in J_0 \setminus \{i\}, \\ t-p_j \geq 0}} x_{ji}^t - \sum_{\substack{j \in J_0 \setminus \{i\}, \\ t+p_i+p_j \leq T}} x_{ij}^{t+p_i} = 0$$

$$(\forall i \in J;\ t = 0, \ldots, T-p_i)$$

The Arc-Time-indexed Formulation

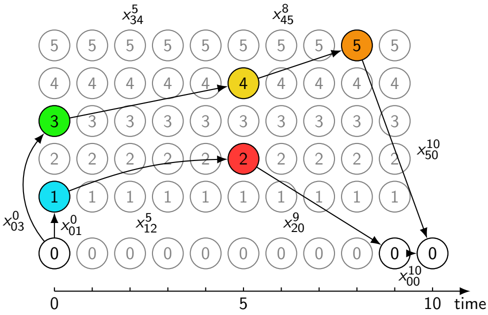


$$\min \sum_{i \in J_0} \sum_{j \in J \setminus \{i\}} \sum_{t=p_i}^{T-p_j} f_j(t+p_j)\, x_{ij}^t$$

$$\text{s.t.} \quad \sum_{i \in J_0 \setminus \{j\}} \sum_{t=p_i}^{T-p_j} x_{ij}^t = 1 \quad (j \in J)$$

$$x \in Z^+$$

ATIF Reformulation



- Pseudo-Schedule: Path from $(0, 0)$ to $(0, T)$ in G
- λ_p: pseudo-schedule p is part of the solution
- $x^t_a = \sum_{p \in P} q^{tp}_a \lambda_p$
- Substituting in the ATIF without flow conservation

ATIF Reformulation

$$\min \quad \sum_{p \in P} \left(\sum_{(i,j)^t \in A} q_{ij}^{tp} f_j(t + p_j) \right) \lambda_p$$

$$\text{s.t.} \quad \sum_{p \in P} \left(\sum_{(j,i)^t \in A} q_{ji}^{tp} \right) \lambda_p = 1 \qquad (\forall\, i \in J) \qquad (\pi_i)$$

$$\sum_{p \in P} \left(\sum_{(0,j)^0 \in A} q_{0j}^{0p} \right) \lambda_p = m \qquad (\pi_0)$$

$$\lambda \geq 0$$

$$\sum_{a^t \in A} \alpha_{al}^t x_a^t \left(\sum_{p \in P} q_a^{tp} \lambda_p \right) \geq b_l \qquad (\forall\, l \in \{n+1, \ldots, r\}) \quad (\pi_l)$$

$$\bar{c}_a^t = f_j(t + p_j) - \sum_{l=0}^{r} \alpha_{al}^t \pi_l$$

Branch-Cut-and-Price

- Pricing
  - Shortest path from $(0, 0)$ to $(0, T)$ in G with arc lengths $\bar{c}_a^t$
- Fixing x_a^t variables by Reduced Costs after every 5 iterations
- Extended Capacity Cuts (Uchoa et al., 2008)
- Dual stabilization of Wentges (1997)
- Strong branching: 8 possible choices
- After Root, if $|A| \leq 200.000$: Feed reduced ATIF to MIP Solver (CPLEX 11.1)

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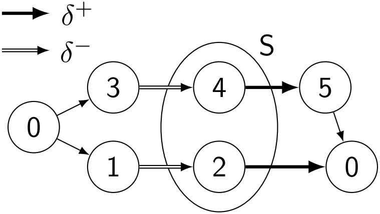
The Overload Elimination Cuts

$$u^t = \sum_{a^t \in \delta^-(S)} x_a^t \quad (t = 1, \ldots, T)$$

$$v^t = \sum_{a^t \in \delta^+(S)} x_a^t \quad (t = 1, \ldots, T)$$



Theorem

For $m \geq 2$, $S \subseteq J$, and $t \in \{1, \ldots, t_{max}\}$:

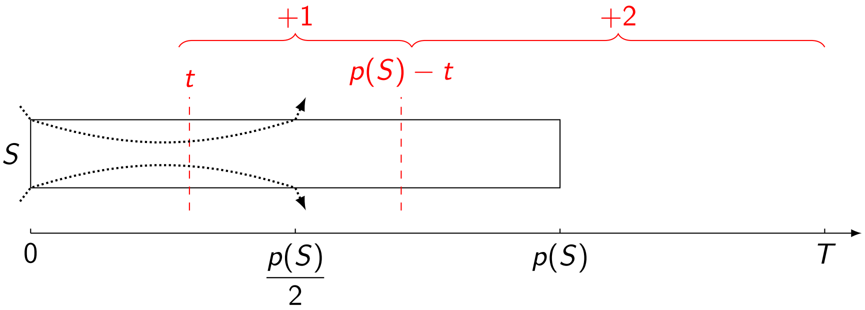
$$\sum_{q=t}^{t_1} v^q + \sum_{q=t_1+1}^{T} 2\, v^q - \sum_{\substack{q=\max\{t_1,\\ T-p(S)+m(t-1)+1\}}}^{T-1} u^q \geq 2,$$

$$t_1 = p(S) - t - (m-2)(t-1).$$

The Overload Elimination Cuts

For two machines:

$$\sum_{q=t}^{p(S)-t} v^q + \sum_{q=p(S)-t+1}^{T} 2\, v^q - \sum_{\substack{q=\max\{p(S)-t,\\ T-p(S)+2t-1\}}}^{T-1} u^q \geq 2$$



OEC Separation

- Genetic Algorithm
- Solution: (S, t)
- $\bar{x} \rightarrow \bar{G}$
- Avg Completion Times: $\bar{C}_j = \sum_{(i,t) | \bar{x}^t_{ji} > 0} t \, \bar{x}^t_{ji}$
- Initial Population
  - connected component of k earliest jobs, including k-th job, $k = 1, \ldots, n$
- Crossover
  - 2 random solutions
  - $S_{child} = S_{father} \cap S_{mother}$
  - $S_{child} = \varnothing \rightarrow$ use a path connecting one element from each
- Local Search: evaluate each single insertion/deletion from S
- Selection: 20 best solutions

Triangle Clique Cuts

- Pessoa, Uchoa and Poggi, 2009 (BCP for the HFVRP)
- $S \subset J, |S| = 3$
- Compatibility graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- $\mathcal{V} = \{(i,j)^t \in A\}$
- $\mathcal{E} = \{((i,j)^t, (j,k)^{t+p_j}) \mid i,j,k \in S\}$

For any independent set $i \in \mathcal{G}$:

$$\sum_{a^t \in I} x_a^t \leq 1$$

Switching to a MIP Solver

- Solve Root Node $\rightarrow$ Branching~~Branching~~ Feed Residual Model to CPLEX
- Residual Model: Excludes variables fixed to zero
- Pessoa et al. (2010): **ATIF** residual model
- Now: **TIF** residual model
- Time-Indexed Formulation, Dyer and Wolsey (1991)
  - Variables y_j^t: job j completes at time t

$$\begin{aligned}
\min \quad & \sum_{j \in J} \sum_{t=p_j}^{T} f_j(t)\, y_j^t \\
\text{s.t.} \quad & \sum_{t=p_j}^{T} y_j^t = 1 && (j \in J) \\
& \sum_{j \in J} \sum_{t'=t}^{\min\{t+p_j-1,\, T\}} y_j^{t'} \le m && (t = 1, \ldots, T) \\
& y \in \{0, 1\}
\end{aligned}$$

Alternative Time-Indexed Formulations

- Variable definition
  - y Variables y_j^t: job j completes at time t
  - z Variables z_j^t: job j completes until time t ($y_j^t = z_j^t - z_j^{t-1}$)
- How to enforce that no more that m machines are running?
  - F Network Flow
  - R Resource Constraints
- 4 different time-indexed formulations (R_y, R_z, F_y, F_z)

Alternative Time-Indexed Formulations

R_y R_z F_y F_z

$$\min \quad \sum_{j \in J} \sum_{t=p_j}^{T} f_j(t)\, y_j^t$$

$$\text{s.t.} \quad \sum_{t=p_j}^{T} y_j^t = 1 \qquad (j \in J)$$

$$\sum_{j \in J} \sum_{t'=t}^{\min\{t+p_j-1,\, T\}} y_j^{t'} \le m \qquad (t = 1, \ldots, T)$$

$$y \in \{0, 1\}$$

Alternative Time-Indexed Formulations

R_y $\quad$ R_z $\quad$ F_y $\quad$ F_z

$$
\begin{aligned}
\min \quad & \sum_{j \in J} \sum_{t=p_j}^{T} f_j(t) \left(z_j^t - z_j^{t-1} \right) \\
\text{s.t.} \quad & z_j^{p_j - 1} = 0 && (j \in J) \\
& z_j^{t-1} \leq z_j^t && (j \in J;\ t = p_j, \ldots, T) \\
& z_j^T = 1 && (j \in J) \\
& \sum_{j \in J} \left(z_j^{\min\{t+p_j-1,\, T\}} - z_j^{t-1} \right) \leq m && (t = 1, \ldots, T) \\
& z \in \{0, 1\}
\end{aligned}
$$

Alternative Time-Indexed Formulations

R_y R_z F_y F_z

$$\min \sum_{j \in J} \sum_{t=p_j}^{T} f_j(t)\, y_j^t$$

$$\text{s.t.} \sum_{t=p_j}^{T} y_j^t = 1 \qquad (j \in J)$$

$$\sum_{j \in J} \sum_{t'=t}^{\min\{t+p_j-1,\, T\}} y_j^{t'} \leq m \qquad (t = 1, \ldots, T)$$

$$y \in \{0, 1\}$$

Alternative Time-indexed Formulations

R_y R_z F_y F_z

$$
\begin{aligned}
\min \quad & \sum_{j \in J} \sum_{t=p_j}^{T} f_j(t)\, y_j^t \\
\text{s.t.} \quad & \sum_{t=p_j}^{T} y_j^t = 1 && (j \in J) \\
& \sum_{j \in J} y_j^{p_j} = m \\
& \sum_{j \in J | t \geq p_j} y_j^t \geq \sum_{j \in J} y_j^{t+p_j} && (t = 1, \ldots, T) \\
& y \in \{0, 1\}
\end{aligned}
$$

Alternative Time-Indexed Formulations

R_y R_z F_y $\underline{F_z}$

$$\min \quad \sum_{j\in J}\sum_{t=p_j}^{T} f_j(t)\left(z_j^t - z_j^{t-1}\right)$$

$$\text{s.t.} \quad z_j^{p_j-1} = 0 \qquad (j \in J)$$

$$z_j^{t-1} \le z_j^t \qquad (j \in J;\ t = p_j, \ldots, T)$$

$$z_j^T = 1 \qquad (j \in J)$$

$$\sum_{j\in J}\left(z_j^{p_j} - z_j^{p_j-1}\right) = m$$

$$\sum_{j\in J | t \ge p_j}\left(z_j^t - z_j^{t-1}\right) \ge \sum_{j\in J}\left(z_j^{t+p_j} - z_j^{t+p_j-1}\right) \qquad (t = 1, \ldots, T)$$

$$z \in \{0,1\}$$

TIF Cuts by Projecting the ATIF Polytope

$$\sum_{i \in J} y_i^t \geq \sum_{j \in J} y_j^{t+p_j} \qquad (t = 1, \ldots, T)$$

$$prec_t(S) = \{i \mid \exists j \in S,\ x_{ij}^t \text{ is not fixed}\}$$

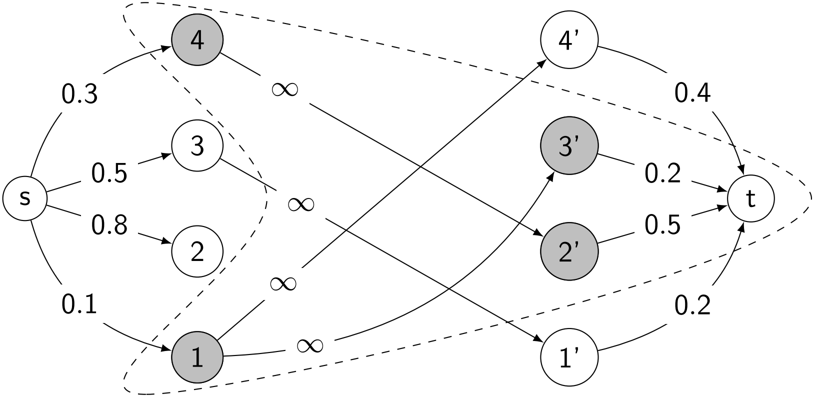
$$\sum_{i \in prec_t(S)} y_i^t \geq \sum_{j \in S} y_j^{t+p_j} \qquad (S \subset J;\ t = 1, \ldots, T)$$

TIF Cuts Separation

$$0.3 + 0.1 < 0.2 + 0.5$$

Violated Cut: $y_1^t + y_4^t \geq y_2^{t+p_2} + y_3^{t+p_3}$

Separation: Minimum Cut



How much of the Integrality Gap is closed?

Table: Root relaxation and cut separation results

		BCP-PMWT		BCP-PMWT-OTI	
n	m	Avg. Gap	Avg. Time	Avg. Gap	Avg. Time
40	2	0.525%	78.0	0.235%	51.9
40	4	0.456%	23.4	0.448%	18.8
50	2	0.379%	256.8	0.276%	193.8
50	4	0.571%	67.8	0.583%	29.9
100	2	0.878%	6297.0	0.114%	3398.8
100	4	0.494%	984.0	0.322%	481.6

Which is the best TIF?

Table: Comparison of Alternative Time-Indexed Formulations

	Average LP Time (s)				Average MIP Time (s)				# Solved			
n	Fy	Ry	Fz	Rz	Fy	Ry	Fz	Rz	Fy	Ry	Fz	Rz
40	0.72	0.84	7.17	0.97	63.17	351.97	122.92	58.28	12	10	12	12
50	1.77	1.98	47.08	2.43	53.46	150.26	70.56	16.47	13	11	14	16

*average times only for the instances solved with all 4 TIFs in up to 3,600 seconds

How much help is the Variable Fixation?

Table: Effect of Variable Fixation in the Rz Time-Index Formulation – Summary

	Average LP Time (s)		Average MIP Time (s)		# Solved	
n	Fix.	w/ Fix.	Fix.	w/ Fix.	Fix.	w/ Fix.
40	0.74	22.54	11.11	561.59	12	10
50	2.04	105.84	11.63	496.61	17	9

*average times only for the instances solved by both in up to 3,600 seconds

How much help are the Projected Cuts?

Table: Effect of Projected Cuts in the Rz Time-Indexed Formulation – Summary

		ATIF	TIF		
n	m	Root Gap	1st LP Gap	Root Gap	Gap Improv.
100	2	0.114%	0.294%	0.249%	16.76%
100	4	0.322%	0.660%	0.646%	11.20%

Overall Results

Table: Full Results - Summary

		BCP-PMWT		BCP-PMWT-OTI	
n	m	# Solved	Avg. Time	# Solved	Avg. Time
40		50	357.9	50	48.1
50		50	5734.9	50	241.9
100	2	18	22523.8	21	7058.5
100	4	16	37667.7	22	5672.0

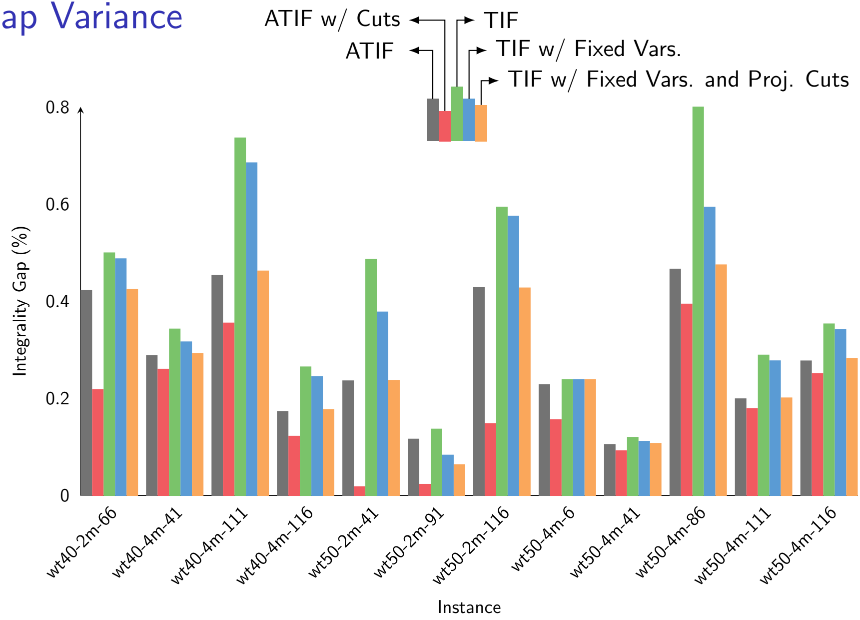
*average times only for the instances solved by both in up to 3,600 seconds

Branching vs Switching to MIP Solver

Table: BCP-PMWT-OTI Best Procedure

		BCP-PMWT				BCP-PMWT-OTI			
n	m	Root	BCP	ATIF MIP	Unsolved	Root	BCP	TIF MIP	Unsolved
40		38	2	10	0	38	1	11	0
50		33	4	13	0	33	3	14	0
100	2	13	2	3	7	16	1	4	4
100	4	7	5	4	9	7	1	14	3

Gap Variance



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Conclusions

- Results
  - 9 instances solved for the first time
  - 84.1% running time decrease for other instances